Shock compression of a perfect gas

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SUMMARY

The compression of a perfect gas between a uniformly moving piston and a rigid wall is discussed in the one-dimensional case. If the piston moves with a finite speed, it will initiate a shock in the gas which will reflect successively from rigid wall and piston and cause the compression process to deviate from a reversible adiabatic process. Expressions are derived for the relative changes in pressure and density at each shock reflection. Then values of density and pressure after any number of shock reflections are computed relative to their initial values, and, in terms of these, the corresponding values of temperature and entropy, as well as shock speeds, are determined. The limiting value of the entropy change, as the number of reflections goes to infinity, is obtained as a function of the ratio of specific heats of the gas and the strength of the initial shock. Hence it is possible to estimate an upper limit to the deviation of the shock compression process from a reversible adiabatic process. Some illustrative numerical examples are given.

I. INTRODUCTION

When a gas is compressed by the motion of a piston, one phenomenon that may cause the compression to deviate from a reversible adiabatic process results from the fact that, if the piston moves with a finite speed, hydrodynamic shocks will be produced. It is the purpose of this paper to discuss, in the idealized case of zero heat conduction and viscosity, the compression of a gas by a piston which moves with finite speed, to give a detailed description of the process as shocks run back and forth between the moving piston and an enclosing rigid wall, and to determine the deviation of this shock compression process from a reversible adiabatic process. In the absence of dissipative effects, the shock front may be represented as a mathematical discontinuity in pressure, density, etc.; for a discussion of this point, see Courant & Friedrichs (1948), also Grad (1952) and Sachs (1946).

Consider the one-dimensional system consisting of a perfect gas with zero viscosity and heat conductivity confined between two plane parallel walls that are impermeable to heat. One wall is considered to be rigid and the other to move toward the first with constant speed. If the speed of the moving wall (or piston) is infinitely slow, then (assuming the internal energy of the gas to be proportional to the temperature) the pressure p and the density ρ will increase continuously according to the adiabatic relation $p\rho^{-\gamma}$ = constant, where γ is the ratio of specific heats, and the process will be reversible. If, however, the piston is suddenly given a finite velocity inward, it will initiate a shock in the gas. The shock will reflect at the rigid wall producing a second shock which will in turn be reflected at the moving wall, and so on. When each new shock passes a given point, the pressure and density of the gas at that point will undergo discontinuous increases. As a shock front traverses the distance between piston and rigid wall, it separates the gas into two regions throughout each of which the pressure, density and material velocity are uniform, but change discontinuously across the shock front. At the instant of reflection at either end, one of the regions has zero volume and the gas is uniform throughout. Since the entropy of the gas increases as each shock passes, the process is irreversible and $p\rho^{-\gamma}$, which is a function of the entropy, no longer remains constant. As a consequence, at a given compression the pressure and temperature of the gas will be higher in the shock compression case than they would be in the case of constant entropy. The amount by which the process deviates from the adiabatic relation $p\rho^{-\gamma}$ = constant depends upon the strength of the initial shock; that is, the ratio of the pressure behind the initial shock to its original value, which in turn is related to the speed of the moving wall.

Using the conservation equations for shocks, it is possible to compute the ratios of density and pressure behind the shock to their values in front of it after each successive reflection. Then the values of density and pressure behind the shock after any number of reflections are computed relative to their initial values, and, in terms of these, the corresponding values of temperature and entropy, as well as the shock speeds, are determined. In order to estimate an upper limit to the deviation from the constant entropy case, the limiting value, as the number of reflections goes to infinity, of the function $p\rho^{-\gamma}$ is computed. This limit is derived as a function of the strength of the original shock and the γ which characterizes the gas. Finally, some illustrative numerical examples are given.

II. Pressure ratio and compression produced by each shock

Let ρ_n, p_n and u_n $(n = 0, 1, 2, \ldots)$ be the density, pressure and material velocity, respectively, behind the nth shock, where the condition $n = 0$ corresponds to the initial conditions in the gas, $n = 1$ corresponds to conditions behind the first shock (that is, the initial shock produced by the moving wall), $n = 2$ corresponds to conditions behind the second shock (that is, the shock produced by the first reflection), and so on. The material velocities, u_n, will be measured relative to the rigid wall and the gas is assumed to be initially at rest; that is, $u_0 = 0$. Let

$$\sigma_n = p_n/p_{n-1}, \tag{1 a}$$

$$\eta_n = \rho_n/\rho_{n-1}. \tag{1 b}$$

The equations describing conservation of mass and momentum may be combined to give the relation*

$$(u_{n-1}-u_n)^2 = (p_n-p_{n-1})(\rho_{n-1}^{-1}-\rho_n^{-1}).$$

The initial and boundary conditions are such that

(*a*) if the shock is produced at the rigid wall,

$$u_n = 0, \qquad u_{n-1} = w;$$

(*b*) if the shock is produced at the moving wall,

$$u_n = w, \qquad u_{n-1} = 0,$$

where w is the velocity of the moving wall. So in either case the left-hand side of the above equation is w^2, and, substituting (1), it becomes

$$w^2\rho_{n-1}\, p_{n-1}^{-1} = (\sigma_n - 1)(1-\eta_n^{-1}). \qquad (2)$$

If the perfect gas has the property that the internal energy is proportional to the temperature, then the internal energy per unit mass is given by

$$E_n = p_n/\rho_n(\gamma-1),$$

and the conservation equations lead to the *Hugoniot relation*

$$\sigma_n = \frac{(\gamma-1)-(\gamma+1)\eta_n}{(\gamma-1)\eta_n-(\gamma+1)} = \frac{1-\mu\eta_n}{\eta_n-\mu}, \qquad (3)$$

where $$\mu = (\gamma+1)/(\gamma-1),$$

and γ is the ratio of specific heats.

To obtain expressions for σ_n and η_n as functions of n, γ and σ_1, first write (2) with n replaced by $n+1$, and divide the resulting equation by (2). The result may be written

$$\sigma_n(\sigma_{n+1}-1)(1-\eta_{n+1}^{-1}) = \eta_n(\sigma_n-1)(1-\eta_n^{-1}).$$

Then, by substituting (3) into this equation, one obtains the equation

$$(\mu\eta_n-1)(\eta_{n+1}-1)^2 = (\eta_n-1)^2(\mu-\eta_{n+1})\eta_{n+1},$$

which is a quadratic having the two solutions:

$$\eta_{n+1} = \eta_n^{-1};$$

and $$\eta_{n+1} = (\mu\eta_n-1)/(\eta_n+\mu-2). \qquad (4)$$

The first of these solutions does not apply because the compression, η_n, must be greater than or equal to unity for all n.

Given the strength of the initial shock (that is, σ_1) and using equation (3) together with the recursion formula (4), it is possible to obtain σ_n and η_n for all n. The general expressions are found to be

$$\sigma_n = (\lambda+\mu+n)/(\lambda+n-1), \qquad (5\text{a})$$

$$\eta_n = (\lambda+\mu+n-1)/(\lambda+n), \qquad (5\text{b})$$

where

$$\lambda = (\mu+1)/(\sigma_1-1), \quad \mu = (\gamma+1)/(\gamma-1), \quad 1 \leqslant \gamma \leqslant \infty, \quad 1 < \sigma_1 \leqslant \infty. \qquad (6)$$

* For a derivation of the shock relations and a general discussion of shock waves in one-dimensional flow see, for example, Courant & Friedrichs (1948), Chapter III, Part C.

That equations (5) do indeed represent the general solution may be verified by induction as follows:

(1) For $n = 1$, (5 b) becomes $\eta_n = (1+\mu\sigma_1)/(\mu+\sigma_1)$, which, on comparison with (3), is seen to be the correct result.

(2) Equation (5 b) satisfies the recursion relation (4).

(3) Equations (5) together satisfy (3).

III. Conditions behind the nth shock

1. *Pressure and density*

Let π_n and κ_n be the pressure and density, respectively, behind the nth shock referred to the initial pressure and density, p_0 and ρ_0. Then, using (1) and (5), one obtains

$$\pi_n = p_n/p_0 = \prod_{i=1}^{n} \sigma_i = \frac{(\lambda+\mu+1)(\lambda+\mu+2)...(\lambda+\mu+n)}{\lambda(\lambda+1)...(\lambda+n-1)}, \tag{7 a}$$

$$\kappa_n = \rho_n/\rho_0 = \prod_{i=1}^{n} \eta_i = \frac{(\lambda+\mu)(\lambda+\mu+1)...(\lambda+\mu+n-1)}{(\lambda+1)(\lambda+2)...(\lambda+n)}. \tag{7 b}$$

On introducing the difference equation for the Γ-function, namely,

$$\Gamma(z+1) = z\Gamma(z), \tag{8}$$

(7) may be written

$$\pi_n = \frac{\Gamma(\lambda)\Gamma(\lambda+\mu+n+1)}{\Gamma(\lambda+\mu+1)\Gamma(\lambda+n)}, \tag{9 a}$$

$$\kappa_n = \frac{\Gamma(\lambda+1)\Gamma(\lambda+\mu+n)}{\Gamma(\lambda+\mu)\Gamma(\lambda+n+1)}. \tag{9 b}$$

2. *Temperature*

Let $\theta_i = p_i/\rho_i$. Then the temperature behind the nth shock referred to its initial value is, from (7),

$$\tau_n = \frac{\theta_n}{\theta_0} = \frac{p_n}{p_0}\frac{\rho_0}{\rho_n} = \frac{\pi_n}{\kappa_n} = \frac{(\lambda+n)(\lambda+\mu+n)}{\lambda(\lambda+\mu)}. \tag{10}$$

3. *Entropy*

Rather than deal with the entropy directly, consider the following function of the entropy (from which, if desired, the entropy can readily be obtained):

$$F(S) = p\rho^{-\gamma} = \exp[(S-S_0')/C_v],$$

where S is the specific entropy, S_0' is a constant and C_v is the specific heat at constant volume which is assumed to be constant. Let $F_i = F(S_i)$. Then the value of this function of the entropy after n reflections, relative to its initial value, is given by

$$\epsilon_n = F_n/F_0 = \exp[(S_n-S_0)/C_v] = (p_n/p_0)(\rho_0/\rho_n)^\gamma$$
$$= \frac{\pi_n}{\kappa_n^\gamma} = \prod_{i=1}^{n} \frac{\sigma}{\eta_i^\gamma}. \tag{11}$$

The substitution of (10) and (9b) into (11) leads to

$$\epsilon_n = \frac{\tau_n}{\kappa_n^{\gamma-1}} = \frac{(\lambda+n)(\lambda+\mu+n)}{\lambda(\lambda+\mu)}\left[\frac{\Gamma(\lambda+\mu)\Gamma(\lambda+n+1)}{\Gamma(\lambda+1)\Gamma(\lambda+\mu+n)}\right]^{\gamma-1}. \quad (12)$$

The limiting value of ϵ_n as $n \to \infty$ will be considered in Section IV.

4. *Shock speed and time*

Let t_n be the time at which the nth reflection occurs, that is, the time at which the $(n+1)$th shock is produced; and let $t_0 = 0$. Then at time t_n the gas has a uniform density equal to ρ_n, and the compression relative to the initial density is $\rho_n/\rho_0 = \kappa_n$. Let L be the initial distance between the piston and rigid wall. Then the distance between piston and rigid wall at time t_n is $L - wt_n$, where w is the speed of the piston, and the compression at time t_n is given by

$$\kappa_n = \frac{L}{L-wt_n}.$$

If $T = L/w$ is the time necessary for the piston to traverse the distance L, the above equation yields

$$t_n = T(1-\kappa_n^{-1}). \quad (13)$$

The time interval between the nth and the $(n+1)$th reflections, that is, the time between the $(n+1)$th and $(n+2)$th shocks, is given by

$$\Delta t_{n+1/2} = t_{n+1} - t_n = T(\kappa_n^{-1} - \kappa_{n+1}^{-2}).$$

From (7b), (1b) and (5b), it follows that

$$\kappa_{n+1} = \kappa_n \rho_{n+1}/\rho_n = \kappa_n \eta_{n+1} = \kappa_n(\lambda+\mu+n)/(\lambda+n+1);$$

and hence

$$\Delta t_{n+1/2} = T\kappa_n^{-1}(\mu-1)/(\lambda+\mu+n), \quad (14)$$

where $\kappa_0 = 1$.

Let D_n be the speed of propagation of the nth shock. If n is odd, the nth shock originates at the moving wall, and

$$D_{n(\mathrm{odd})} = \frac{L-wt_{n-1}}{\Delta t_{n-1/2}}.$$

On the substitution of (13), (14) and $L = wT$, this equation becomes

$$D_{n(\mathrm{odd})} = w[(\lambda+n)/(\mu-1)+1].$$

If n is even, the nth shock orginates at the rigid wall, and

$$D_{n(\mathrm{even})} = \frac{L-wt_{n-1}-w\Delta t_{n-1/2}}{\Delta t_{n-1/2}}.$$

In the same way as above, this equation becomes

$$D_{n(\mathrm{even})} = w(\lambda+n)/(\mu-1).$$

The alternative expressions may be combined in the form

$$D_n = w\left(\frac{\lambda+n}{\mu-1}+a\right), \qquad a = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases} \quad (15)$$

IV. Limiting value of the entropy change

Let ϵ_∞^0 be the limiting value of ϵ_n as the number of reflections becomes infinite. Then, from (11),

$$\epsilon_\infty^0(\gamma, \sigma_1) = \lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \frac{F_n}{F_0} = \prod_{i=1}^{\infty} \frac{\sigma_i}{\eta_i^\gamma}, \tag{16}$$

and, by the substitution of (12),

$$\epsilon_\infty^0(\gamma, \sigma_1) = \frac{1}{\lambda(\lambda+\mu)}\left[\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda+1)}\right]^{\gamma-1} \lim_{n\to\infty}\left\{(\lambda+n)(\lambda+\mu+n)\left[\frac{\Gamma(\lambda+n+1)}{\Gamma(\lambda+\mu+n)}\right]^{\gamma-1}\right\}.$$

By the introduction of the asymptotic formula for the gamma function

$$\Gamma(z) \sim (2\pi)^{1/2} e^{-z} z^{z-1/2}, \tag{17}$$

and by the use of (6) and (8), the limit on the right of the above expression becomes

$$e^{\gamma+1} \lim_{n\to\infty} [(\lambda+n)/(\lambda+\mu+n)]^{(\gamma-1)n} = 1,$$

where the final limit is evaluated by taking the logarithm and applying l'Hospital's rule. Hence,

$$\epsilon_\infty^0(\gamma, \sigma_1) = \frac{1}{\lambda(\lambda+\mu)}\left[\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda+1)}\right]^{\gamma-1}. \tag{18}$$

This expression may be generalized to give the limiting value of $F(S)$ measured relative to its value behind any shock, say the kth, where k is finite. Thus let

$$\epsilon_\infty^k(\gamma, \sigma_1) = \lim_{n\to\infty} \frac{F_n}{F_k} = \prod_{i=k+1}^{\infty} \frac{\sigma_i}{\eta_i^\gamma}, \quad k = 0, 1, 2, \ldots. \tag{19}$$

Then, using (5), one obtains the recursion formula

$$\epsilon_\infty^k = \frac{\eta_k^\gamma}{\sigma_k} \prod_{i=k}^{\infty} \frac{\sigma_i}{\eta_i^\gamma} = \frac{\eta_k^\gamma}{\sigma_k} \epsilon_\infty^{k-1} = \frac{(\lambda+k-1)(\lambda+\mu+k-1)^\gamma}{(\lambda+\mu+k)(\lambda+k)^\gamma} \epsilon_\infty^{k-1},$$

from which, together with (18), it is readily demonstrated by induction that

$$\epsilon_\infty^k(\gamma, \sigma_1) = \frac{1}{(\lambda+k)(\lambda+\mu+k)}\left[\frac{\Gamma(\lambda+\mu+k)}{\Gamma(\lambda+k+1)}\right]^{\gamma-1}. \tag{20}$$

It is of interest to obtain the limiting values of $\epsilon_\infty^k(\gamma, \sigma_1)$ corresponding to the extreme values of γ. Thus, as $\gamma \to 1$, then $\mu \to \infty$, $\lambda \to \infty$; and, because of (17) and (6),

$$\epsilon_\infty^k(1, \sigma_1) = \lim_{\gamma\to 1} \epsilon_\infty^k(\gamma, \sigma_1) = \lim_{\mu\to\infty} e^{-2\mu/(\mu-1)}[(\lambda+\mu+k)/(\lambda+k)]^{(2\lambda+2k+\mu)/(\mu-1)}$$
$$= e^{-2} \sigma_1^{(\sigma_1+1)/(\sigma_1-1)} \tag{21}$$

which is seen to be independent of k. In the case where $\gamma \to \infty$, then $\mu \to 1$, $\lambda \to \lambda_0 = 2/(\sigma_1 - 1)$; and by taking the logarithm of the second factor in (20) and expanding it into a power series in $(\mu - 1)$, one obtains

$$\epsilon_\infty^k(\infty, \sigma_1) = \lim_{\gamma\to\infty} \epsilon_\infty^k(\gamma, \sigma_1) = \frac{1}{(\lambda_0+k)(\lambda_0+k+1)} \exp\left[\frac{2\Gamma'(\lambda_0+k+1)}{\Gamma(\lambda_0+k+1)}\right]. \tag{22}$$

(Tables of the function $\psi(z) = \Gamma'(z)/\Gamma(z)$ are given in Davis (1933).)

In the limiting case where the initial shock is a 'strong shock'* (that is, where $\sigma_1 \to \infty$ and $\eta_1 \to \mu$), it is seen from (6) that $\lambda \to 0$. Hence, the preceding formulae may be reduced to those applying to the special case where the initial shock is a 'strong shock' by setting $\lambda = 0$. It is seen that (20) becomes infinite for this case if either $k = 0$ or $\gamma = 1$, but otherwise remains finite.

V. Numerical examples

In order to compare the case of compression at constant entropy with the process of shock compression discussed in the foregoing sections, let the same symbols as used above, but without subscripts, represent corresponding quantities in the case of constant entropy. Then the pressure π, the density κ and the temperature τ, referred to initial values, are related by the expressions

$$\pi\kappa^{-\gamma} = 1, \qquad \pi\kappa^{-1} = \tau.$$

The corresponding relations for conditions behind the nth shock are

$$\pi_n \kappa_n^{-\gamma} = \epsilon_n, \qquad \pi_n \kappa_n^{-1} = \tau_n.$$

Thus, with the same initial conditions and the same compression $\kappa = \kappa_n$, the relation between the pressures that would be arrived at by the two processes is

$$p_n/p = \pi_n/\pi = \epsilon_n;$$

and, since temperature is proportional to pressure at a fixed density,

$$\theta_n/\theta = \tau_n/\tau = \epsilon_n.$$

As the number of reflections becomes large, ϵ_n approaches the limiting value $\epsilon_\infty^0(\gamma, \sigma_1)$. Hence, for a given γ and σ_1, the possible values of π_n and κ_n are confined to a strip in the (π, κ^{-1}) plane (PV diagram) which is bounded by the two adiabatics $\pi\kappa^{-\gamma} = 1$ and $\pi\kappa^{-\gamma} = \epsilon_\infty^0(\gamma, \sigma_1)$. The case of a diatomic gas ($\gamma = 1{\cdot}4$) is illustrated in figure 1 for two different values of the strength of the initial shock, $\sigma_1 = 5$ and $\sigma_1 = 50$. The figure represents a PV diagram on a logarithmic scale, where π and κ^{-1} are pressure and specific volume expressed in units of the initial pressure and specific volume, p_0 and ρ_0^{-1}, respectively. The points (π_n, κ_n), plotted as circled dots, are seen to be converging quite rapidly to the adiabatics for the limiting values of the entropy. In table 1 numerical values of $\epsilon_\infty^0(\gamma, \sigma_1)$ are listed for several different values of γ and σ_1.

For a given γ and σ_1, the greatest relative change in $F(S) = p\rho^{-\gamma}$ caused by the various shocks occurs in the first shock. It is of interest to see to what extent the process of shock compression subsequent to the first shock will deviate from a constant-entropy process that starts from conditions corresponding to those produced by the first shock. In the limiting case where the first shock is a 'strong shock' ($\sigma_1 = \infty$), the first shock causes

* The special case where the initial shock is a 'strong shock' has been discussed previously in an unpublished report by the present authors (1953).

an infinite change in entropy, but the subsequent change as the number of reflections goes to infinity remains finite for $\gamma > 1$. In table 2 numerical values of $\epsilon^1_\infty(\gamma, \sigma_1)$ are listed for several different values of γ and σ_1. In figure 2 a *PV* diagram similar to that of figure 1 is shown, where now π and κ^{-1} are pressure and specific volume expressed in units of the pressure and specific volume behind the first shock, p_1 and ρ_1^{-1}, respectively. The case illustrated is that of a 'strong shock', which, for a given γ, gives the greatest deviation from constant entropy.

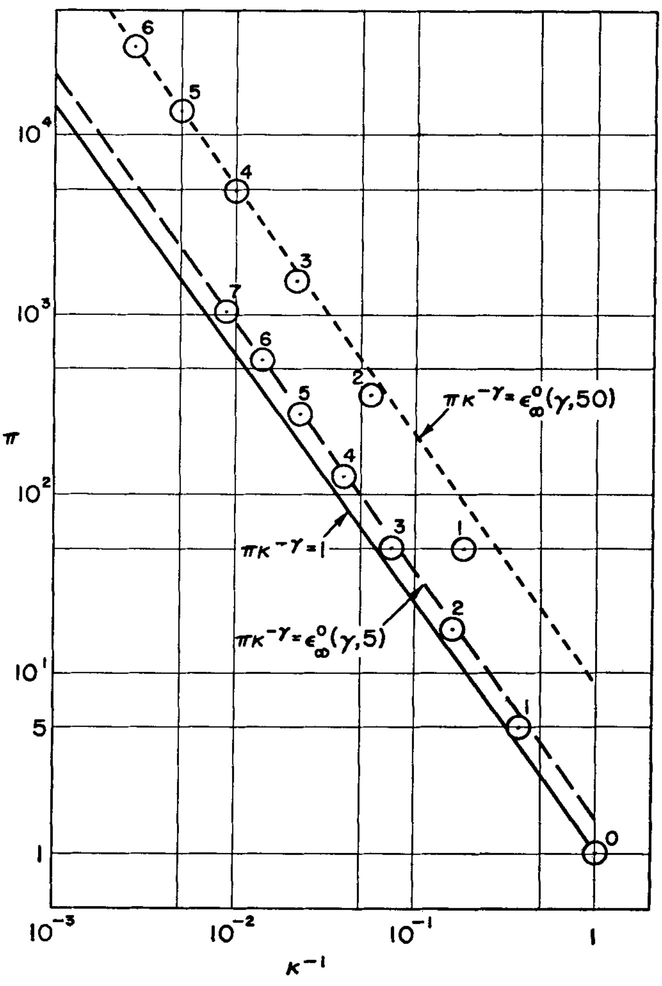

Figure 1. *PV* diagram for $\gamma = 1{\cdot}4$, where π and κ^{-1} are pressure and specific volume in units of their initial values. The straight lines are adiabatics for the initial value of entropy (solid) and the limiting values of entropy (dashed) in the two cases : $\sigma_1 = 5$; $\sigma_1 = 50$. The circled dots represent values of pressure and specific volume behind the *n*th shock referred to their initial values, where the numbers indicate values of *n* in each case.

σ_1 \ γ	1	1·4	5/3	2	3	∞
2	1·083	1·077	1·074	1·071	1·067	1·055
5	1·513	1·511	1·506	1·500	1·485	1·434
10	2·26	2·31	2·32	2·31	2·28	2·18
25	4·42	4·72	4·78	4·81	4·76	4·52
50	7·94	8·75	8·88	8·94	8·91	8·45
100	14·85	16·83	17·14	17·27	17·25	16·33

Table 1. ϵ^0_∞ as a function of γ, the ratio of specific heats of the gas, and σ_1, the pressure ratio in the original shock. ϵ^0_∞ is the value of the ratio of $p\rho^{-\gamma}$ after an infinite number of reflections to its initial value, where p and ρ are pressure and density, respectively.

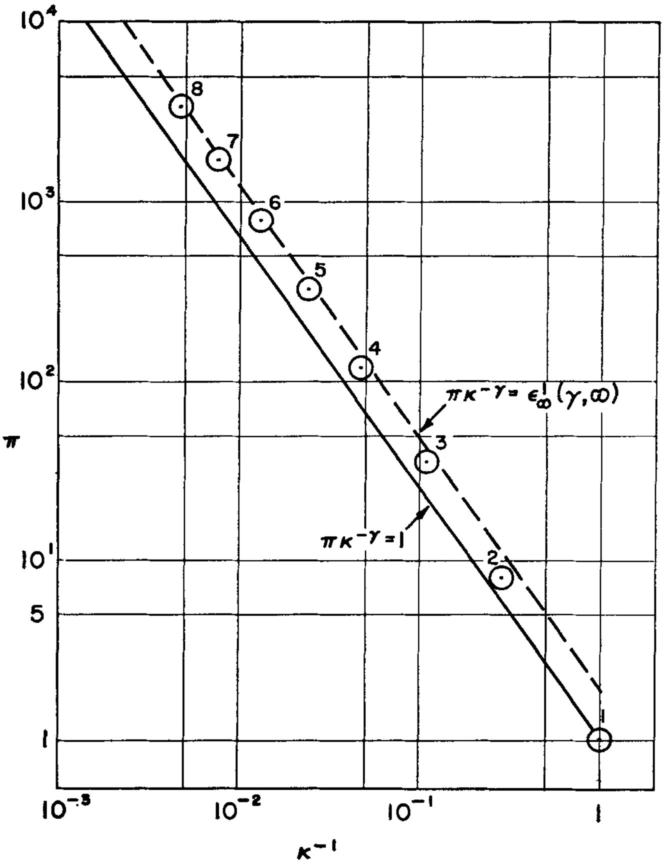

Figure 2. *PV* diagram for $\gamma = 1{\cdot}4$, where π and κ^{-1} are pressure and specific volume in units of their values behind the first shock. The straight lines are adiabatics for the value of entropy behind first shock (solid) and limiting value of entropy for the case $\sigma_1 = \infty$, i.e. a 'strong shock'. The circled dots represent values of pressure and specific volume behind the *n*th shock referred to their values behind the first shock, where the numbers indicate values of *n*.

σ_1 \ γ	1	1·4	5/3	2	3	∞
2	1·083	1·062	1·056	1·050	1·042	1·028
5	1·51	1·29	1·24	1·20	1·15	1·09
10	2·26	1·50	1·39	1·31	1·22	1·12
25	4·42	1·73	1·53	1·42	1·28	1·14
50	7·94	1·84	1·59	1·45	1·31	1·15
100	14·85	1·91	1·63	1·48	1·32	1·16
∞	∞	1·99	1·66	1·50	1·33	1·16

Table 2. ϵ^1_∞ as a function of γ, the ratio of specific heats of the gas, and σ_1, the pressure ratio in the original shock. ϵ^1_∞ is the value of the ratio of $p\rho^{-\gamma}$ after an infinite number of reflections to its value behind the first shock, where p and ρ are pressure and density, respectively.

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